

Yarn-Level Simulation of Woven Cloth

Supplementary Document: Forces and Jacobians

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This document defines in detail forces and Jacobians for internal yarn forces and inter-yarn contact. All notation refers to Fig.2 in the main document, and force terms are described for yarns passing through the yarn crossing \mathbf{q}_0 . We also rewrite the energy potentials for better readability.

Let us first define some auxiliary quantities. The norm l_i of the vector connecting \mathbf{q}_0 and each of its adjacent crossing nodes \mathbf{q}_i :

$$l_i = \|\mathbf{x}_i - \mathbf{x}_0\|. \quad (1)$$

The unit vector \mathbf{d}_i connecting \mathbf{q}_0 and each of its adjacent crossing nodes \mathbf{q}_i :

$$\mathbf{d}_i = \frac{\mathbf{x}_i - \mathbf{x}_0}{l_i}. \quad (2)$$

The vector \mathbf{w}_i between \mathbf{q}_0 and each of its adjacent crossing nodes \mathbf{q}_i , normalized by arc length:

$$\mathbf{w}_i = \frac{\mathbf{x}_i - \mathbf{x}_0}{\|u_i - u_0\|}. \quad (3)$$

It is also convenient to define the projection to the normal plane of \mathbf{d}_i :

$$\mathbf{P}_i = \mathbf{I} - \mathbf{d}_i \mathbf{d}_i^T. \quad (4)$$

1 Stretch

Energy of the warp segment $[\mathbf{q}_0, \mathbf{q}_1]$, assuming $\Delta u = u_1 - u_0 > 0$:

$$V = \frac{1}{2} k_s \Delta u (\|\mathbf{w}_1\| - 1)^2. \quad (5)$$

Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1} = -\mathbf{F}_{\mathbf{x}_0} = -k_s (\|\mathbf{w}_1\| - 1) \mathbf{d}_1. \quad (6)$$

Forces on warp coordinates:

$$F_{u_1} = -F_{u_0} = \frac{1}{2} k_s (\|\mathbf{w}_1\|^2 - 1). \quad (7)$$

Non-zero Jacobians:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = \frac{k}{l_1} \mathbf{P}_1 - \frac{k}{\Delta u} \mathbf{I}, \quad (8)$$

$$\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_0}}{\partial u_0} = -\frac{\partial F_{u_1}}{\partial u_0} = -\frac{\partial F_{u_0}}{\partial u_1} = -k_s \frac{\|\mathbf{w}_1\|^2}{\Delta u}, \quad (9)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_0} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = k_s \frac{\|\mathbf{w}_1\|}{\Delta u} \mathbf{d}_1, \quad (10)$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_1} = \frac{\partial F_{u_0}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_1}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_0}}{\partial \mathbf{x}_1} = \frac{k_s}{\Delta u} \mathbf{w}_1^T. \quad (11)$$

2 Bending

Bending angle θ between warp segments $[\mathbf{q}_2, \mathbf{q}_0]$ and $[\mathbf{q}_0, \mathbf{q}_1]$, assuming $u_1 > u_0 > u_2$:

$$\theta = \arccos \left(-\mathbf{d}_1^T \mathbf{d}_2 \right). \quad (12)$$

Energy of straight yarn:

$$V = k_b \frac{\theta^2}{u_1 - u_2}. \quad (13)$$

Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1} = -\frac{2 k_b \theta}{l_1 (u_1 - u_2) \sin \theta} \mathbf{P}_1 \mathbf{d}_2, \quad (14)$$

$$\mathbf{F}_{\mathbf{x}_2} = -\frac{2 k_b \theta}{l_2 (u_1 - u_2) \sin \theta} \mathbf{P}_2 \mathbf{d}_1, \quad (15)$$

$$\mathbf{F}_{\mathbf{x}_0} = -(\mathbf{F}_{\mathbf{x}_1} + \mathbf{F}_{\mathbf{x}_2}). \quad (16)$$

Forces on warp coordinates:

$$F_{u_1} = -F_{u_2} = \frac{k_b \theta^2}{(u_1 - u_2)^2}, \quad F_{u_0} = 0. \quad (17)$$

Non-zero Jacobians:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{2 k_b}{l_1^2 (u_1 - u_2) \sin \theta} \left(\theta \left(\mathbf{P}_1 \mathbf{d}_2 \mathbf{d}_1^T + \frac{\cos \theta}{\sin^2 \theta} \mathbf{P}_1 \mathbf{d}_2 \mathbf{d}_2^T \mathbf{P}_1 + \cos \theta \mathbf{P}_1 + \mathbf{d}_1 \mathbf{d}_2^T \mathbf{P}_1 \right) - \frac{1}{\sin \theta} \mathbf{P}_1 \mathbf{d}_2 \mathbf{d}_2^T \mathbf{P}_1 \right), \quad (18)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2} = -\frac{2 k_b}{l_2 l_1 (u_1 - u_2) \sin \theta} \left(\theta \left(\mathbf{P}_1 - \frac{\cos \theta}{\sin^2 \theta} \mathbf{P}_1 \mathbf{d}_2 \mathbf{d}_1^T \right) + \frac{1}{\sin \theta} \mathbf{P}_1 \mathbf{d}_2 \mathbf{d}_1^T \right) \mathbf{P}_2, \quad (19)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1} = -\frac{2 k_b}{l_1 l_2 (u_1 - u_2) \sin \theta} \left(\theta \left(\mathbf{P}_2 - \frac{\cos \theta}{\sin^2 \theta} \mathbf{P}_2 \mathbf{d}_1 \mathbf{d}_2^T \right) + \frac{1}{\sin \theta} \mathbf{P}_2 \mathbf{d}_1 \mathbf{d}_2^T \right) \mathbf{P}_1, \quad (20)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2} = \frac{2 k_b}{l_2^2 (u_1 - u_2) \sin \theta} \left(\theta \left(\mathbf{P}_2 \mathbf{d}_1 \mathbf{d}_2^T + \frac{\cos \theta}{\sin^2 \theta} \mathbf{P}_2 \mathbf{d}_1 \mathbf{d}_1^T \mathbf{P}_2 + \cos \theta \mathbf{P}_2 + \mathbf{d}_2 \mathbf{d}_1^T \mathbf{P}_2 \right) - \frac{1}{\sin \theta} \mathbf{P}_2 \mathbf{d}_1 \mathbf{d}_1^T \mathbf{P}_2 \right), \quad (21)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2} \right), \quad (22)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_1} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_2} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_2} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_2} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial \mathbf{x}_0} \right), \quad (23)$$

$$\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_2}}{\partial u_2} = -\frac{\partial F_{u_1}}{\partial u_2} = -\frac{\partial F_{u_2}}{\partial u_1} = -\frac{2 k_b \theta^2}{(u_1 - u_2)^3}, \quad (24)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_2} = \frac{2 k_b \theta}{l_1 (u_1 - u_2)^2 \sin \theta} \mathbf{P}_1 \mathbf{d}_2, \quad (25)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_2} = \frac{2 k_b \theta}{l_2 (u_1 - u_2)^2 \sin \theta} \mathbf{P}_2 \mathbf{d}_1, \quad (26)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_1} = -\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial u_2} = -\left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial u_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_2}}{\partial u_1} \right), \quad (27)$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_1} = -\frac{\partial F_{u_2}}{\partial \mathbf{x}_1} = \frac{2 k_b \theta}{l_1 (u_1 - u_2)^2 \sin \theta} \mathbf{d}_2^T \mathbf{P}_1, \quad (28)$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_2} = -\frac{\partial F_{u_2}}{\partial \mathbf{x}_2} = \frac{2 k_b \theta}{l_2 (u_1 - u_2)^2 \sin \theta} \mathbf{d}_1^T \mathbf{P}_2, \quad (29)$$

$$\frac{\partial F_{u_1}}{\partial \mathbf{x}_0} = -\frac{\partial F_{u_2}}{\partial \mathbf{x}_0} = -\left(\frac{\partial F_{u_1}}{\partial \mathbf{x}_1} + \frac{\partial F_{u_1}}{\partial \mathbf{x}_2} \right). \quad (30)$$

3 Shear

Shear angle ϕ between warp segment $[\mathbf{q}_0, \mathbf{q}_1]$ and weft segment $[\mathbf{q}_0, \mathbf{q}_3]$, assuming $u_1 > u_0$ and $v_3 > v_0$:

$$\phi = \arccos(\mathbf{d}_1^T \mathbf{d}_3). \quad (31)$$

Energy:

$$V = \frac{1}{2} k_x L \left(\phi - \frac{\pi}{2} \right)^2. \quad (32)$$

Forces on warp and weft coordinates are all zero. Forces on crossing points:

$$\mathbf{F}_{\mathbf{x}_1} = \frac{k_x L \left(\phi - \frac{\pi}{2} \right)}{l_1 \sin \phi} \mathbf{P}_1 \mathbf{d}_3, \quad (33)$$

$$\mathbf{F}_{\mathbf{x}_3} = \frac{k_x L \left(\phi - \frac{\pi}{2} \right)}{l_3 \sin \phi} \mathbf{P}_3 \mathbf{d}_1, \quad (34)$$

$$\mathbf{F}_{\mathbf{x}_0} = -(\mathbf{F}_{\mathbf{x}_1} + \mathbf{F}_{\mathbf{x}_3}). \quad (35)$$

Non-zero Jacobians:

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} = \frac{k_x L}{l_1^2 \sin \phi} \left(\left(\phi - \frac{\pi}{2} \right) \left(-\mathbf{P}_1 \mathbf{d}_3 \mathbf{d}_1^T + \frac{\cos \phi}{\sin^2 \phi} \mathbf{P}_1 \mathbf{d}_3 \mathbf{d}_3^T \mathbf{P}_1 - \cos \phi \mathbf{P}_1 - \mathbf{d}_1 \mathbf{d}_3^T \mathbf{P}_1 \right) - \frac{1}{\sin \phi} \mathbf{P}_1 \mathbf{d}_3 \mathbf{d}_3^T \mathbf{P}_1 \right), \quad (36)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} = \frac{k_x L}{l_3 l_1 \sin \phi} \left(\left(\phi - \frac{\pi}{2} \right) \left(\frac{\cos \phi}{\sin^2 \phi} \mathbf{P}_1 \mathbf{d}_3 \mathbf{d}_1^T + \mathbf{P}_1 \right) - \frac{1}{\sin \phi} \mathbf{P}_1 \mathbf{d}_3 \mathbf{d}_1^T \right) \mathbf{P}_3, \quad (37)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} = \frac{k_x L}{l_1 l_3 \sin \phi} \left(\left(\phi - \frac{\pi}{2} \right) \left(\frac{\cos \phi}{\sin^2 \phi} \mathbf{P}_3 \mathbf{d}_1 \mathbf{d}_3^T + \mathbf{P}_3 \right) - \frac{1}{\sin \phi} \mathbf{P}_3 \mathbf{d}_1 \mathbf{d}_3^T \right) \mathbf{P}_1, \quad (38)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} = \frac{k_x L}{l_3^2 \sin \phi} \left(\left(\phi - \frac{\pi}{2} \right) \left(-\mathbf{P}_3 \mathbf{d}_1 \mathbf{d}_3^T + \frac{\cos \phi}{\sin^2 \phi} \mathbf{P}_3 \mathbf{d}_1 \mathbf{d}_1^T \mathbf{P}_3 - \cos \phi \mathbf{P}_3 - \mathbf{d}_3 \mathbf{d}_1^T \mathbf{P}_3 \right) - \frac{1}{\sin \phi} \mathbf{P}_3 \mathbf{d}_1 \mathbf{d}_1^T \mathbf{P}_3 \right), \quad (39)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} = - \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0} = - \left(\frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} \right), \quad (40)$$

$$\frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_1} = - \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_1} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_1} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_3} = - \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_3} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_3} \right), \quad \frac{\partial \mathbf{F}_{\mathbf{x}_0}}{\partial \mathbf{x}_0} = - \left(\frac{\partial \mathbf{F}_{\mathbf{x}_1}}{\partial \mathbf{x}_0} + \frac{\partial \mathbf{F}_{\mathbf{x}_3}}{\partial \mathbf{x}_0} \right). \quad (41)$$

4 Parallel Contact

Energy of the warp segment $[\mathbf{q}_0, \mathbf{q}_1]$, assuming contact exists, i.e., $\Delta u = u_1 - u_0 < d$:

$$V_{0,1} = \frac{1}{2} k_c L (\Delta u - d)^2. \quad (42)$$

Forces on warp coordinates:

$$F_{u_0} = -F_{u_1} = k_c L (\Delta u - d). \quad (43)$$

Non-zero Jacobians:

$$\frac{\partial F_{u_0}}{\partial u_0} = \frac{\partial F_{u_1}}{\partial u_1} = -\frac{\partial F_{u_0}}{\partial u_1} = -\frac{\partial F_{u_1}}{\partial u_0} = -k_c L. \quad (44)$$