Yarn-Level Simulation of Woven Cloth

Supplementary Document: Forces and Jacobians

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This document defines in detail forces and Jacobians for internal yarn forces and inter-yarn contact. All notation refers to Fig.2 in the main document, and force terms are described for yarns passing through the yarn crossing \( q_0 \). We also rewrite the energy potentials for better readability.

Let us first define some auxiliary quantities. The norm \( l_i \) of the vector connecting \( q_0 \) and each of its adjacent crossing nodes \( q_i \):

\[
l_i = \| x_i - x_0 \|. \tag{1}
\]

The unit vector \( d_i \) connecting \( q_0 \) and each of its adjacent crossing nodes \( q_i \):

\[
d_i = \frac{x_i - x_0}{l_i}. \tag{2}
\]

The vector \( w_i \) between \( q_0 \) and each of its adjacent crossing nodes \( q_i \), normalized by arc length:

\[
w_i = \frac{x_i - x_0}{\| u_i - u_0 \|}. \tag{3}
\]

It is also convenient to define the projection to the normal plane of \( d_i \):

\[
P_i = I - d_i d_i^T. \tag{4}
\]

1 Stretch

Energy of the warp segment \([q_0, q_1]\), assuming \( \Delta u = u_1 - u_0 > 0 \):

\[
V = \frac{1}{2} k_s \Delta u (\| w_1 \| - 1)^2. \tag{5}
\]

Forces on crossing points:

\[
F_{x_1} = -F_{x_0} = -k_s (\| w_1 \| - 1) d_1. \tag{6}
\]

Forces on warp coordinates:

\[
F_{u_1} = -F_{u_0} = \frac{1}{2} k_s (\| w_1 \|^2 - 1). \tag{7}
\]

Non-zero Jacobians:

\[
\frac{\partial F_{x_1}}{\partial x_1} = \frac{\partial F_{x_0}}{\partial x_0} = \frac{\partial F_{x_1}}{\partial x_0} = \frac{\partial F_{x_0}}{\partial x_1} = \frac{k}{l_1} \frac{P_1}{\Delta u} - \frac{k}{\Delta u} I, \tag{8}
\]

\[
\frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_0}}{\partial u_0} = \frac{\partial F_{u_1}}{\partial u_0} = \frac{\partial F_{u_0}}{\partial u_1} = -k_s \frac{\| w_1 \|^2}{\Delta u}, \tag{9}
\]

\[
\frac{\partial F_{x_1}}{\partial u_1} = \frac{\partial F_{x_0}}{\partial u_0} = \frac{\partial F_{x_1}}{\partial u_0} = \frac{\partial F_{x_0}}{\partial u_1} = k_s \frac{\| w_1 \|}{\Delta u} d_1, \tag{10}
\]

\[
\frac{\partial F_{u_1}}{\partial x_1} = \frac{\partial F_{u_0}}{\partial x_0} = \frac{\partial F_{u_1}}{\partial x_0} = \frac{\partial F_{u_0}}{\partial x_1} = \frac{k_s}{\Delta u} w_1^T. \tag{11}
\]

2 Bending

Bending angle \( \theta \) between warp segments \([q_2, q_0]\) and \([q_0, q_1]\), assuming \( u_1 > u_0 > u_2 \):

\[
\theta = \arccos \left( -d_1^T d_2 \right). \tag{12}
\]

Energy of straight yarn:

\[
V = k_b \frac{\theta^2}{u_1 - u_2}. \tag{13}
\]
Forces on crossing points:

\[ F_{x_1} = -\frac{2 k_b \theta}{l_1 (u_1 - u_2) \sin \theta} P_1 d_2, \]  
\[ F_{x_2} = -\frac{2 k_b \theta}{l_2 (u_1 - u_2) \sin \theta} P_2 d_1, \]  
\[ F_{x_0} = -(F_{x_1} + F_{x_2}). \]  

Forces on warp coordinates:

\[ F_{u_1} = -F_{u_2} = \frac{k_b \theta^2}{(u_1 - u_2)^T}, \]
\[ F_{u_0} = 0. \]  

Non-zero Jacobians:

\[ \frac{\partial F_{x_1}}{\partial x_1} = \frac{2 k_b}{l_1^2 (u_1 - u_2) \sin \theta} \left( \theta \left( P_1 d_2 d_1^T + \frac{\cos \theta}{\sin^2 \theta} P_1 d_2 d_2^T P_1 + \cos \theta P_1 + d_1 d_2^T P_1 \right) - \frac{1}{\sin \theta} P_1 d_2 d_2^T P_1 \right), \]  
\[ \frac{\partial F_{x_1}}{\partial x_2} = -\frac{2 k_b}{l_2 l_1 (u_1 - u_2) \sin \theta} \left( \theta \left( P_1 - \frac{\cos \theta}{\sin^2 \theta} P_1 d_1^T \right) + \frac{1}{\sin \theta} P_1 d_2^T \right) P_2, \]  
\[ \frac{\partial F_{x_2}}{\partial x_1} = -\frac{2 k_b}{l_1 l_2 (u_1 - u_2) \sin \theta} \left( \theta \left( P_2 - \frac{\cos \theta}{\sin^2 \theta} P_2 d_1^T \right) + \frac{1}{\sin \theta} P_2 d_1^T \right) P_1, \]  
\[ \frac{\partial F_{x_2}}{\partial x_2} = \frac{2 k_b}{l_1^2 (u_1 - u_2) \sin \theta} \left( \theta \left( P_2 d_1 d_1^T + \frac{\cos \theta}{\sin^2 \theta} P_2 d_2 d_2^T P_2 + \cos \theta P_2 + d_2 d_1^T P_2 \right) - \frac{1}{\sin \theta} P_2 d_1 d_2^T P_2 \right), \]  
\[ \frac{\partial F_{x_1}}{\partial x_0} = -\left( \frac{\partial F_{x_1}}{\partial x_1} + \frac{\partial F_{x_1}}{\partial x_2} \right), \]
\[ \frac{\partial F_{x_2}}{\partial x_0} = -\left( \frac{\partial F_{x_2}}{\partial x_1} + \frac{\partial F_{x_2}}{\partial x_2} \right), \]
\[ \frac{\partial F_{x_0}}{\partial x_1} = -\left( \frac{\partial F_{x_0}}{\partial x_1} + \frac{\partial F_{x_0}}{\partial x_2} \right), \]
\[ \frac{\partial F_{x_0}}{\partial x_0} = \left( \frac{\partial F_{x_0}}{\partial x_1} + \frac{\partial F_{x_0}}{\partial x_2} \right), \]
\[ \frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_2}}{\partial u_2} = -\frac{\partial F_{u_2}}{\partial u_1} = -\frac{\partial F_{u_2}}{\partial u_1} = -\frac{2 k_b \theta^2}{(u_1 - u_2)^T}, \]  
\[ \frac{\partial F_{x_1}}{\partial u_1} = \frac{2 k_b \theta}{l_1 (u_1 - u_2)^2 \sin \theta} P_1 d_2, \]  
\[ \frac{\partial F_{x_2}}{\partial u_1} = \frac{2 k_b \theta}{l_2 (u_1 - u_2)^2 \sin \theta} P_2 d_1, \]  
\[ \frac{\partial F_{x_0}}{\partial u_1} = \left( \frac{\partial F_{x_0}}{\partial u_1} + \frac{\partial F_{x_0}}{\partial u_2} \right), \]
\[ \frac{\partial F_{u_1}}{\partial u_1} = \frac{2 k_b \theta}{l_1 (u_1 - u_2)^2 \sin \theta} d_2^T P_1, \]  
\[ \frac{\partial F_{u_2}}{\partial u_2} = \frac{2 k_b \theta}{l_2 (u_1 - u_2)^2 \sin \theta} d_1^T P_2, \]  
\[ \frac{\partial F_{u_0}}{\partial u_0} = -\left( \frac{\partial F_{u_0}}{\partial x_1} + \frac{\partial F_{u_0}}{\partial x_2} \right). \]

3 Shear

Shear angle \( \phi \) between warp segment \([q_0, q_1]\) and weft segment \([q_0, q_3]\), assuming \( u_1 > u_0 \) and \( v_3 > v_0 \):

\[ \phi = \arccos(d_3^T d_3). \]  

Energy:

\[ V = \frac{1}{2} k_s L \left( \frac{\phi - \pi}{2} \right)^2. \]  

Forces on warp and weft coordinates are all zero. Forces on crossing points:

\[ F_{x_1} = \frac{k_s L (\phi - \pi/2)}{l_1 \sin \phi} P_1 d_3, \]  
\[ F_{x_2} = \frac{k_s L (\phi - \pi/2)}{l_3 \sin \phi} P_3 d_1, \]  
\[ F_{x_0} = -(F_{x_1} + F_{x_2}). \]
Non-zero Jacobians:

\[
\frac{\partial F_{x_1}}{\partial x_1} = \frac{k_c L}{l_1^2 \sin \phi} \left( \frac{\phi - \pi}{2} \left( -P_1 d_3 d_1^T + \frac{\cos \phi}{\sin^2 \phi} P_1 d_3 d_1^T P_1 - \frac{\cos \phi}{\sin \phi} P_1 - d_1 d_3^T P_1 \right) - \frac{1}{\sin \phi} P_1 d_3 d_3^T P_1 \right),
\]

\[
\frac{\partial F_{x_3}}{\partial x_3} = \frac{k_c L}{l_3 l_1 \sin \phi} \left( \frac{\phi - \pi}{2} \left( \frac{\cos \phi}{\sin^2 \phi} P_1 d_3 d_1^T + P_1 \right) - \frac{1}{\sin \phi} P_1 d_3 d_1^T P_3, \right)
\]

\[
\frac{\partial F_{x_1}}{\partial x_1} = \frac{k_c L}{l_1^2 \sin \phi} \left( \frac{\phi - \pi}{2} \left( -P_3 d_1 d_3^T + \frac{\cos \phi}{\sin^2 \phi} P_3 d_1 d_1^T P_3 - \frac{\cos \phi}{\sin \phi} P_3 - d_3 d_1^T P_3 \right) - \frac{1}{\sin \phi} P_3 d_1 d_3^T P_3 \right),
\]

\[
\frac{\partial F_{x_3}}{\partial x_3} = -\left( \frac{\partial F_{x_1}}{\partial x_1} + \frac{\partial F_{x_1}}{\partial x_3} \right), \quad \frac{\partial F_{x_1}}{\partial x_1} = -\left( \frac{\partial F_{x_3}}{\partial x_3} + \frac{\partial F_{x_3}}{\partial x_1} \right),
\]

\[
\frac{\partial F_{x_0}}{\partial x_0} = -\left( \frac{\partial F_{x_1}}{\partial x_1} + \frac{\partial F_{x_1}}{\partial x_3} \right), \quad \frac{\partial F_{x_1}}{\partial x_1} = -\left( \frac{\partial F_{x_3}}{\partial x_3} + \frac{\partial F_{x_3}}{\partial x_1} \right), \quad \frac{\partial F_{x_0}}{\partial x_0} = -\left( \frac{\partial F_{x_1}}{\partial x_1} + \frac{\partial F_{x_3}}{\partial x_3} \right).
\]

4 Parallel Contact

Energy of the warp segment [q₀, q₁], assuming contact exists, i.e., Δu = u₁ - u₀ < d:

\[
V_{0,1} = \frac{1}{2} k_c L (\Delta u - d)^2.
\]

Forces on warp coordinates:

\[
F_{u_0} = -F_{u_1} = k_c L (\Delta u - d).
\]

Non-zero Jacobians:

\[
\frac{\partial F_{u_0}}{\partial u_0} = \frac{\partial F_{u_1}}{\partial u_1} = \frac{\partial F_{u_0}}{\partial u_1} = \frac{\partial F_{u_1}}{\partial u_0} = -k_c L.
\]